

The University of Chicago

THE BEHAVIOR OF THE HESSIAN
AT A MULTIPLE POINT
OF A CURVE

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1936

By
JAMES EDWARD CASE, S.J.

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THE BEHAVIOR OF THE HESSIAN AT A MULTIPLE POINT OF A CURVE

I. HISTORICAL NOTES

The discoverer of the curve with which we shall be dealing was Dr. Otto Hesse, who in an article (1)* written in 1844 defined a new covariant of a quantic. This covariant has since been known as the Hessian of the quantic. Hesse, besides defining the covariant, found also that when the quantic is a form in three homogeneous variables, $f(xyz)$, the covariant Hessian, $H(xyz)$, has the property that the curve whose equation is $H = 0$ passes through the inflexion points of the curve whose equation is $f = 0$.

In a memorable paper Steiner (2) dealt with this curve from the synthetic viewpoint. Following in his footsteps Kötter (3) and Cremona (4) used the synthetic approach.

Clebsch (5) considered the curve analytically and established the Plücker numbers and most of the properties of the Hessian of the cubic and of the quartic.

Brill (6) showed the correspondence between a curve and its Hessian at inflexion points. He also showed that in general when the curve C has a k -fold point at A , the Hessian of C has a $(3k - 4)$ -fold point at A .

Wölffing (7) stated analytically some of the simpler relationships between the k -fold point A on C and the corresponding point A on H .

*The numbers in parentheses refer to the bibliography at the end of the paper.

Segre (8) built on Brill's work, and added further the conditions necessary that the point A on H corresponding to the k-fold point A on C be of multiplicity $3k - 3$, $3k - 2$, or $3k - 1$.

The following facts may be stated as common knowledge:

a. The Hessian $H = 0$ of a curve $f = 0$ of degree n is of degree $3(n - 2)$.

b. The Hessian of a curve passes through all the inflexion points and all the multiple points of the curve.

c. A curve and its Hessian meet in $3n(n - 2)$ points, lying one at each inflexion point, six at each node, and eight at each simple cusp of C.

d. When a curve has an ordinary cusp at A, its Hessian will have a triple point at A made up of a cusp and an ordinary branch. Its cuspidal tangent coincides with the cuspidal tangent to the curve at A.

e. When a curve has a node at A, its Hessian will have a node at A with the same nodal tangents as C.

f. When a curve has a k-fold point ($1 < k < n$) at A, its Hessian will have at A a point of multiplicity at least $3k - 4$.

g. If not all the tangents to C at A are coincident the multiplicity of the point A on H will be exactly $3k - 4$, and the tangents to H at A will be the k tangents to C at A, plus the $2(k - 2)$ lines which comprise the Hessian group of these k tangents, given by:

$$\begin{vmatrix} \partial^2 u_k / \partial x^2 & \partial^2 u_k / \partial x \partial y \\ \partial^2 u_k / \partial x \partial y & \partial^2 u_k / \partial y^2 \end{vmatrix} = 0,$$

where the symbol u_k has the same meaning as in equation (1) on page 4.

II. THE PROBLEM

It is the purpose of this paper to give a complete analysis of the relationship between the general k -fold point ($1 < k < n$) at A of a curve C and its corresponding point A of H . By use of a new way of writing the equation of the Hessian and by means of the Newton Diagram (to be defined in the next paragraph), the nature of the point A of H , corresponding to the point A of C , can be determined; in fact the roots are separated and the substitutions which lead to the Puiseux expansions are obtained.

III. THE NEWTON POLYGON AND THE NEWTON DIAGRAM

Let $f(xy) = 0$ be written

$$\sum c_{ab}(x - h)^a(y - k)^b = 0.$$

According to the method developed by Newton of studying the form of a curve at the point (h, k) , the exponents a and b of $x - h$ and $y - k$ are considered as number pairs and as such are plotted on the Cartesian plane.

By the Newton Polygon (N. P.) of a curve is meant that convex configuration made up of the totality of lines determined by the above points such that none of the points lies to the left and below these lines.

To determine the form of a curve at a point (h, k) the N. P. suffices (10) but to determine the form of the Hessian of C at the same point it is necessary to define what will be called the Newton Diagram (N. D.) of the curve at the point.

By the Newton Diagram of a curve at a point is meant the following configuration:

1. The sides of the N. P.,

2. The lines of slope - 1 given by the terms of equal degree in the non-homogeneous equation $f(xy) = 0$.

3. Certain additional lines determined by the points (a, b) ; in particular it will include that configuration which will be known as the Second Newton Polygon, to be defined on page 16.

IV. VARIOUS FORMS IN WHICH $f = 0$ MAY BE WRITTEN, AND SOME CONSEQUENCES

Given an irreducible plane curve C with a k -fold point at the origin. The equation of C will be given in any one of the following forms:

A. Homogeneous Form:

$$(1) \quad f(xyz) = 0 = u_k z^{n-k} + u_{k+1} z^{n-(k+1)} + \dots + u_{k+\alpha_j} z^{n-(k+\alpha_j)} \\ + \dots + u_{n-1} z + u_n,$$

where $u_{k+\alpha_j}$ is a homogeneous polynomial of degree $k + \alpha_j$ in x and y , where $j = 0, 1, \dots, n - k$, and $\alpha_j = j$.

B. Non-Homogeneous Form:

$$(2) \quad f(xy) = 0 = u_k + u_{k+1} + \dots + u_{k+\alpha_j} + \dots + u_{n-1} + u_n,$$

where $u_{k+\alpha_j}$ has the same meaning as above.

C. A Form which Exhibits the Possible Factorization of the Polynomials:

$$(3) \quad f(xy) = 0 = x^{p_0} y^{f_0} \varphi^{n_0} + \dots + x^{p_j} y^{f_j} \varphi^{n_j} + \dots,$$

where each φ^{n_j} (for $j = 0, 1, \dots, n$) represents a homogeneous polynomial of degree $k + \alpha_j = p_j + f_j$ in x and y , which does not vanish for $x = 0$ or for $y = 0$. It is to be noted that because of the assumed irreducibility of C , not every p_j can be different

from zero, and not every ξ_j can be different from zero. We shall indicate by ρ_n the first ρ which is zero, and by ξ_p the first ξ which is zero.

Consider the points on the N. D. of $f = 0$ which correspond to the terms of

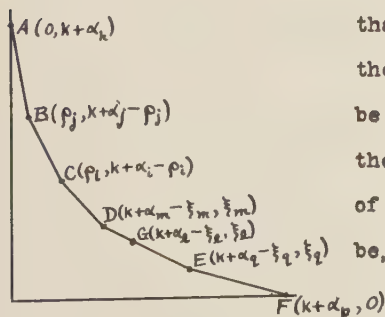
$$u_{k+\alpha_j} = x^{\rho_j} y^{\xi_j} \cdot \sum_{\ell=0}^{k+\alpha_j-\rho_j-\xi_j} c_{\ell} x^{k+\alpha_j-\rho_j-\xi_j-\ell} y^{\ell},$$

in which ρ_j, ξ_j are integers ≥ 0 , and $\rho_j + \xi_j \leq k + \alpha_j$. These points form a line segment L_j whose slope is -1 . Thus we have in the N. D. of $f = 0$ a set of parallel line segments L , one for each power of z appearing in equation (1).

The correctness of the following statements is apparent.

1. For $\rho_j = 0$, there is one point of L_j on the y -axis. Its coordinates are $(0, k + \alpha_j)$.
2. For $\xi_j = 0$, there is one point of L_j on the x -axis. Its coordinates are $(k + \alpha_j, 0)$.
3. For $\rho_j \neq 0$, the point on L_j with the least abscissa is given by $(\rho_j, k + \alpha_j - \rho_j)$. It will be referred to as the ρ -end point of the line segment.
4. For $\xi_j \neq 0$, the point on L_j with the least ordinate is given by $(k + \alpha_j - \xi_j, \xi_j)$. It will be referred to as the ξ -end point of the line segment.

Consider a side of the N. P. of $f = 0$ whose slope is less



than -1 , for example, the side CB in the accompanying figure. On it will be found (at least) two points. Let the coordinates of the ρ -end points of the segments determining this side be, say, $(\rho_1, k + \alpha_1 - \rho_1)$ and

$(\beta_j, k + \alpha_j - \beta_j)$. The slope of the side is then

$$= \frac{\alpha_j - \alpha_i + \beta_i - \beta_j}{\beta_i - \beta_j},$$

and the branches of C, in number $|\alpha_j - \alpha_i + \beta_i - \beta_j|$, which correspond to this side of the N. P. of $f = 0$, are separated by means of the substitutions

$$\begin{cases} x = t^{\alpha_j - \alpha_i + \beta_i - \beta_j} \\ y = ut^{\beta_i - \beta_j} \end{cases}$$

Again, consider a side of the N. P. of $f = 0$ whose slope is greater than -1, for example the side ED in the figure. On it will also be found (at least) two points with coordinates, say, $(k + \alpha_m - \beta_m, \beta_m)$ and $(k + \alpha_l - \beta_l, \beta_l)$.

The slope of this side is given by:

$$= \frac{\beta_m - \beta_l}{\alpha_l - \alpha_m + \beta_m - \beta_l}$$

and the branches of C, in number $|\beta_m - \beta_l|$, which correspond to this side of the N. P. of $f = 0$ are separated by means of the substitution:

$$\begin{cases} x = t^{\beta_m - \beta_l} \\ y = ut^{\alpha_l - \alpha_m + \beta_m - \beta_l} \end{cases}$$

V. A NEW FORM FOR THE EQUATION OF THE HESSIAN

Given the equation of a curve, $f(xyz) = 0$. Then the equation of the Hessian of this curve is:

$$H = 0 = \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}.$$

Each element of the above determinant is the sum of a finite number of terms, since f is a polynomial, e.g.,

$$\begin{aligned} f_{xx} &= (u_k)_{xx} + (u_{k+1})_{xx} + \dots \\ f_{xy} &= (u_k)_{xy} + (u_{k+1})_{xy} + \dots \end{aligned}$$

Accordingly, $H = 0$ may be expressed as a sum of determinants. We adopt the notation

$$\begin{aligned} a_1 &= (n - k - \alpha_1)(n - k - \alpha_1 - 1)u_{k+\alpha_1} \\ d_1 &= (n - k - \alpha_1)(\partial/\partial x)u_{k+\alpha_1} \\ e_1 &= (n - k - \alpha_1)(\partial/\partial y)u_{k+\alpha_1} \\ b_1 &= (\partial^2/\partial x^2)u_{k+\alpha_1} \\ g_1 &= (\partial^2/\partial x \partial y)u_{k+\alpha_1} \\ c_1 &= (\partial^2/\partial y^2)u_{k+\alpha_1} \end{aligned} \quad (4)$$

with corresponding definitions for $a_m, d_m, \dots, c_r$.

Then we find that $H = 0$ may be expressed in the following form:

$$H = 0 = \sum_{s=0}^{3n-3k-2} \begin{vmatrix} a_1 & d_m & e_r \\ d_1 & b_m & g_r \\ e_1 & g_m & c_r \end{vmatrix} z^{3(n-2)-(3k-4+s)}, \quad (5)$$

or, for brevity,

$$H = 0 = \sum_{s=0}^{3n-3k-2} (imr) z^{3(n-2)-(3k-4+s)}, \quad (6)$$

where i, m, r are integers positive or zero, and $i + m + r = s$.

It will be noted that each of the determinants (imr) is a homogeneous polynomial in x and y of degree $3k + \alpha_1 + \alpha_m + \alpha_r - 4$ according to definitions (4).

The expansion of (6) when z is set equal to one, is:

$$\begin{aligned}
 H = 0 = & [(000)] + [(001) + (010) + (100)] + [(002) + (011) \\
 & + (020) + (101) + (110) + (200)] + [(003) + \dots] + \dots \\
 & + [\dots + (imr) + \dots] + [(n-1, n-1, n) \\
 & + (n-1, n, n-1)],
 \end{aligned}$$

where, for example,

$$(001) = \begin{vmatrix} (n-k)(n-k-1)u_k & (n-k)\partial u_k/\partial x & (n-k-1)\partial u_{k+1}/\partial y \\ (n-k)\partial u_k/\partial x & \partial^2 u_k/\partial x^2 & \partial^2 u_{k+1}/\partial x \partial y \\ (n-k)\partial u_k/\partial y & \partial^2 u_k/\partial x \partial y & \partial^2 u_{k+1}/\partial y^2 \end{vmatrix}.$$

N. B. Since, by definitions (4), $a_{n-1} = a_n = d_n = e_n = 0$, the determinants $(n, n-1, n-1)$, $(n-1, n, n)$, $(n, n-1, n)$, $(n, n, n-1)$, (n, n, n) all vanish.

The term $u_{k+\alpha_1}$ appearing in equation (2) may be written out as follows:

$$\begin{aligned}
 (7) \quad u_{k+\alpha_1} = & c_{\rho_1} x^{\rho_1} y^{k+\alpha_1-\rho_1} + c_{\rho_{1j}} x^{\rho_{1j}} y^{k+\alpha_1-\rho_{1j}} + c_{\rho_{1p}} x^{\rho_{1p}} y^{k+\alpha_1-\rho_{1p}} \\
 & + \dots + c_{k+\alpha_1-\xi_{1q}} x^{k+\alpha_1-\xi_{1q}} y^{\xi_{1q}} + c_{k+\alpha_1-\xi_{1l}} x^{k+\alpha_1-\xi_{1l}} y^{\xi_{1l}} \\
 & + c_{k+\alpha_1-\xi_1} x^{k+\alpha_1-\xi_1} y^{\xi_1}.
 \end{aligned}$$

Similar expressions for $u_{k+\alpha_m}$ and $u_{k+\alpha_r}$ are obtained by replacing 1 by m, and i by r in (7). In equation (7) ρ_1 and ξ_1 are the lowest powers of x and y in the respective terms, and hence either or both may be zero. The same may be said of ρ_m , ξ_m , ρ_r , and ξ_r .

Again ρ_{1j} , ξ_{1l} ; ρ_{mj} , ξ_{ml} ; ρ_{rj} , ξ_{rl} are never taken to be zero, but always greater than or equal to one, while ρ_{1p} , ξ_{1q} ; ρ_{mp} , ξ_{mq} ; ρ_{rp} , ξ_{rq} are never taken to be zero or one, but always greater than or equal to two, in such a fashion that if, for example, ρ_1 chances to be 1, in accord with the above convention, ρ_{1j} will also be taken = 1, and again if ξ_m chances to

be = 2, ξ_{mq} will be taken as equal to $\xi_m\chi$ and this to be also equal to ξ_m .

If (3) be kept in mind, definitions (4) may now be written:

$$\begin{aligned}
 (8) \quad a_1 &= x^{\rho_i} y^{\xi_i} \phi^{k+\alpha_i-\rho_i-\xi_i} \\
 d_1 &= x^{\rho_{ij}-1} y^{\xi_i} \phi^{k+\alpha_i-\rho_{ij}-\xi_i} \\
 e_1 &= x^{\rho_i} y^{\xi_{ie}-1} \phi^{k+\alpha_i-\rho_i-\xi_{ie}} \\
 b_1 &= x^{\rho_{ip}-2} y^{\xi_i} \phi^{k+\alpha_i-\rho_{ip}-\xi_i} \\
 g_1 &= x^{\rho_{ij}-1} y^{\xi_{ie}-1} \phi^{k+\alpha_i-\rho_{ij}-\xi_{ie}} \\
 c_1 &= x^{\rho_i} y^{\xi_{iq}-2} \phi^{k+\alpha_i-\rho_i-\xi_{iq}}
 \end{aligned}$$

The coefficient of $z^{3(n-2)-(3k-4+s)}$ in (6), where $s = 1 + m + r$, is the sum of the six determinants: $(imr) + (irm) + (rim) + (rmi) + (mir) + (mri)$. This sum will hereafter be designated by the symbol $[imr]$.

VI. ANALYSIS OF $[imr]$

It is clear that $[imr]$, as well as each of its members, is homogeneous in x and y and of degree $3k + \alpha_1 + \alpha_m + \alpha_r - 4$. Since they are homogeneous, the points corresponding to them will lie on the same line segment L of the N. D. of $H = 0$. We have but to determine the ρ -end point and the ξ -end point of this line.

Because of the equations (4), (7), (8) and the conventions on the ρ and ξ symbols, $[imr]$ is manifestly (notationally) symmetric in ρ and ξ , for we have as distinct factors in $[imr]$:

	<u>in the x's</u>	<u>in the y's</u>
	$\rho_i + \rho_m + \rho_{rp} - 2$	$\xi_i + \xi_m + \xi_{rq} - 2$
	$\rho_i + \rho_{mp} + \rho_r - 2$	$\xi_i + \xi_{mq} + \xi_r - 2$
	$\rho_{ip} + \rho_m + \rho_r - 2$	$\xi_{iq} + \xi_m + \xi_r - 2$
(9)	$\rho_i + \rho_{mj} + \rho_{rj} - 2$	$\xi_i + \xi_{m\chi} + \xi_{r\chi} - 2$
	$\rho_{ij} + \rho_m + \rho_{rj} - 2$	$\xi_{i\chi} + \xi_m + \xi_{r\chi} - 2$
	$\rho_{ij} + \rho_{mj} + \rho_r - 2$	$\xi_{i\chi} + \xi_{m\chi} + \xi_r - 2$

Accordingly, what is said of the ρ 's may be said with equal validity of the ξ 's, if the ξ 's are subjected to the same conditions as are the ρ 's. Hence all that has to be done is to find the coordinates of the ρ -end points of the various line segments.

A cursory glance at (9) where merely the exponents are shown will reveal the fact that if any one of ρ_i, ρ_m, ρ_r is ≥ 2 , the ρ -end point of the line segment of degree $3k + \alpha_i + \alpha_m + \alpha_r - 4$ of the N. D. of $H = 0$ will be

$$(\rho_i + \rho_m + \rho_r - 2, 3k + \alpha_i + \alpha_m + \alpha_r - \rho_i - \rho_m - \rho_r - 2).$$

Similarly, if any one of ξ_i, ξ_m, ξ_r is ≥ 2 , the ξ -end point of the line segment will be

$$(3k + \alpha_i + \alpha_m + \alpha_r - \xi_i - \xi_m - \xi_r - 2, \xi_i + \xi_m + \xi_r - 2).$$

Example: Let $u_{k+\alpha_i} = x^2y^3\phi^{k+\alpha_i-5}$ and $u_{k+\alpha_h} = \phi^{k+\alpha_h}$. Here $\rho_i = 2, \rho_h = 0; \xi_i = 3, \xi_h = 0$. Then [ihh] will give as ρ -end point: $(0, 3k + \alpha_i + 2\alpha_h - 4)$, and as ξ -end point: $(3k + \alpha_i + 2\alpha_h - 5, 1)$.

Again it will be seen that if any two of ρ_i, ρ_m, ρ_r are ≥ 1 , say, $\rho_i = \rho_j \geq 1; \rho_m = \rho_n \geq 1$, then $(\rho_i + \rho_m + \rho_r - 2, 3k + \alpha_i + \alpha_m + \alpha_r - \rho_i - \rho_m - \rho_r - 2)$ will be the ρ -end point.

Likewise, if any two of ξ_i, ξ_m, ξ_r are ≥ 1 , say $\xi_m = \xi_n \geq 1; \xi_r = \xi_s \geq 1$, then $(3k + \alpha_i + \alpha_m + \alpha_r - \xi_i - \xi_m - \xi_r - 2, \xi_i + \xi_m + \xi_r - 2)$ will be the ξ -end point.

Example: Let $u_{k+\alpha_i} = x\phi^{k+\alpha_i-1}; u_{k+\alpha_m} = xy\phi^{k+\alpha_m-2}; u_{k+\alpha_r} = y\phi^{k+\alpha_r-1}$. Here $\rho_i = 1, \rho_m = 1, \rho_r = 0; \xi_i = 0, \xi_m = 1, \xi_r = 1$. Then [imr] will give as ρ -end point: $(0, 3k + \alpha_i + \alpha_m + \alpha_r - 4)$, and as ξ -end point: $(3k + \alpha_i + \alpha_m + \alpha_r - 4, 0)$.

The above results show that we have only to settle the question as to what will be the ρ -end points and the ξ -end points in the case when two of the ρ 's or of the ξ 's are zero, and the remaining one equal to one or zero.

1. Let, for example, $\rho_i = \rho_{ij} = 1$; $\rho_m = \rho_r = 0$. Then $[1mr]$ will have as ρ -end point that one of the following which has the least abscissa:

$$\begin{aligned} &(\rho_{ip} - 2, \quad 3k + \alpha_i + \alpha_m + \alpha_r - \rho_{ip} - 2), \\ &(\rho_{ij} + \rho_{mj} - 2, \quad 3k + \alpha_i + \alpha_m + \alpha_r - \rho_{ij} - \rho_{mj} - 2), \\ &(\rho_{ij} + \rho_{rj} - 2, \quad 3k + \alpha_i + \alpha_m + \alpha_r - \rho_{ij} - \rho_{rj} - 2). \end{aligned}$$

It is clear that since $\rho_{ij} = 1$, if one of ρ_{mj} or ρ_{rj} equals 1, the ρ -end point will be $(0, 3k + \alpha_i + \alpha_m + \alpha_r - 4)$.

2. Let $\rho_i = \rho_m = \rho_r = 0$. Then, if at least two of ρ_{ij} , ρ_{mj} , ρ_{rj} equal 1, $[1mr]$ will have as ρ -end point: $(0, 3k + \alpha_i + \alpha_m + \alpha_r - 4)$.

But if fewer than two of ρ_{ij} , ρ_{mj} , ρ_{rj} exist and have values less than 2 and at the same time there exists at least one of ρ_{ip} , ρ_{mp} , ρ_{rp} with value greater than or equal to 2, $[1mr]$ will have as ρ -end point that one of the following which has the least abscissa:

$$\begin{aligned} &(\rho_{ip} - 2, \quad 3k + \alpha_i + \alpha_m + \alpha_r - \rho_{ip} - 2), \\ &(\rho_{mp} - 2, \quad 3k + \alpha_i + \alpha_m + \alpha_r - \rho_{mp} - 2), \\ &(\rho_{rp} - 2, \quad 3k + \alpha_i + \alpha_m + \alpha_r - \rho_{rp} - 2). \end{aligned}$$

VII. METHOD OF ATTACK

The statement of our problem has been given, viz., to analyze completely the relationship between the k -fold point A on $f = 0$ and its corresponding point A on $H = 0$.

A new form of writing $H = 0$ has been given as a sum of the determinants (5) or (6).

A typical determinant as given by $[1mr]$ has been analyzed, and the ρ -end point for all situations has been found.

Our Method of Attack on the problem will be to institute a comparison between the N. P. of $f = 0$ and the N. P. of $H = 0$, and by means of this comparison to arrive finally at the substitutions separating the branches of $H = 0$ corresponding to those of $f = 0$. This comparison will be effected by the use of definitions (4) and (6) together with the findings of Section VI.

VIII. DATA

Let the origin of coordinates be moved to the singular point in question. The work of the construction of the sides of the N. P. of $H = 0$ which correspond to the sides of the N. P. of $f = 0$ will be divided into two cases:

Case A. There may be sides of the N. P. of $f = 0$ neither of whose end points has a coordinate equal to zero.

Case B. There may be sides of the N. P. of $f = 0$ with one end point having a zero coordinate. If the abscissa is zero, the polygon begins at the y-axis at the point $(0, k + \alpha_k)$, which will be called the ρ -closing point of the polygon. If the ordinate is zero, the polygon ends at the point $(k + \alpha_p, 0)$, called the ξ -closing point of the polygon.

Case A. If we mark all the points contributed by all the possible determinants of (6) going to make up $H = 0$, found by letting s of (6) take all the values within its range, i.e., from $s = 0$ to $s = 3n - 3k - 2$, we shall find that certain groups of determinants, e.g., $[111]$, $[11j]$, $[1jj]$, $[jjj]$ made from combinations

of $u_{k+\alpha_1}$ and $u_{k+\alpha_j}$ will have as ρ -end points points which lie on a line, and similarly will have as ξ -end points points which lie on a line.

For $[i i i]$ the ρ -end point is $(3\rho_1 - 2, 3k + 3\alpha_1 - 3\rho_1 - 2)$, while for $[i i j]$ the ρ -end point is $(2\rho_1 + \rho_j - 2, 3k + 2\alpha_1 + \alpha_j - 2\rho_1 - \rho_j - 2)$, and for $[i j j]$ the ρ -end point is $(\rho_1 + 2\rho_j - 2, 3k + \alpha_1 + 2\alpha_j - \rho_1 - 2\rho_j - 2)$, and for $[j j j]$ the ρ -end point is $(3\rho_j - 2, 3k + 3\alpha_j - 3\rho_j - 2)$. It is evident that these ρ -end points lie on a line whose slope is evidently the same as the slope of the side of the N. P. of $f = 0$ determined by the points: $(\rho_1, k + \alpha_1 - \rho_1)$ and $(\rho_j, k + \alpha_j - \rho_j)$. If we connect the ρ -end points of all of these line segments we shall have part of a polygon which corresponds in number of sides and slope of sides to the various sides of the N. P. of $f = 0$, for all sides whose determining ρ 's and ξ 's are greater than zero.

This polygon will constitute a part of the N. P. of $H = 0$ if and only if no other determinant or set of determinants contributes points which lie to the left and below the end points produced by these determinants.

Let $(\rho_{\chi}, k + \alpha_{\chi} - \rho_{\chi})$ be a point on the N. D. of $f = 0$ such that it does not lie to the left and below the side of the N. P. of $f = 0$ determined by $(\rho_1, k + \alpha_1 - \rho_1)$ and $(\rho_j, k + \alpha_j - \rho_j)$.

The necessary and sufficient condition that $(\rho_{\chi}, k + \alpha_{\chi} - \rho_{\chi})$ be such a point may be expressed

$$\frac{\alpha_{\ell} - \alpha_1}{\alpha_j - \alpha_1} \geq \frac{\rho_1 - \rho_{\ell}}{\rho_1 - \rho_j},$$

while similar conditions that $(\rho_m, k + \alpha_m - \rho_m)$ and $(\rho_q, k + \alpha_q - \rho_q)$ be such points are had by substituting m for χ and q for χ in the above condition. If we add these three inequalities we have:

$$(10) \quad \frac{(\alpha_l + \alpha_m + \alpha_q) - 3\alpha_i}{\alpha_j - \alpha_i} \geq \frac{3\rho_i - (\rho_l + \rho_m + \rho_q)}{\rho_i - \rho_j}$$

Now consider the set of determinants $[\chi_{mq}]$. The ρ -end point (of this line segment of degree $3k + \alpha_l + \alpha_m + \alpha_q - 4$) is:

$$(\rho_l + \rho_m + \rho_q - 2, 3k + \alpha_l + \alpha_m + \alpha_q - \rho_l - \rho_m - \rho_q - 2).$$

The slope of the line connecting the ρ -end points of $[iii]$, ..., $[jjj]$ is:

$$- \frac{\alpha_j - \alpha_i + \rho_i - \rho_j}{\rho_i - \rho_j},$$

whereas the slope of the line connecting the ρ -end points of the line segment determined by $[iii]$ and $[\chi_{mq}]$ is:

$$- \frac{3\rho_i - (\rho_l + \rho_m + \rho_q) + (\alpha_l + \alpha_m + \alpha_q) - 3\alpha_i}{3\rho_i - (\rho_l + \rho_m + \rho_q)}.$$

Therefore the ρ -end point of $[\chi_{mq}]$ will not lie to the left and below the points on the side of the N. P. of $H = 0$ determined by $[iii]$, ..., $[jjj]$ if and only if:

$$- \frac{3\rho_i - (\rho_l + \rho_m + \rho_q) + (\alpha_l + \alpha_m + \alpha_q) - 3\alpha_i}{3\rho_i - (\rho_l + \rho_m + \rho_q)} \leq - \frac{\alpha_j - \alpha_i + \rho_i - \rho_j}{\rho_i - \rho_j}$$

But this inequality may be written:

$$\frac{(\alpha_l + \alpha_m + \alpha_q) - 3\alpha_i}{\alpha_j - \alpha_i} \geq \frac{3\rho_i - (\rho_l + \rho_m + \rho_q)}{\rho_i - \rho_j},$$

and accordingly is seen to be identically (10).

Therefore, we have proved the very useful

THEOREM: If $(\rho_l, k + \alpha_l - \rho_l)$, $(\rho_m, k + \alpha_m - \rho_m)$, $(\rho_q, k + \alpha_q - \rho_q)$ are points which do not lie to the left and below any side of the N. P. of $f = 0$, determined, say, by $(\rho_i, k + \alpha_i - \rho_i)$ and $(\rho_j, k + \alpha_j - \rho_j)$, where all the ρ 's considered are greater than zero, then no determinant or set of determinants $[\chi\chi\chi]$, ..., $[\mu\mu\mu]$, ..., $[\eta\eta\eta]$ in any combination, will produce

points which lie to the left and below the line on which lie the points corresponding to the determinants [111], ..., [jjj].

COROLLARY 1. No term of $H = 0$, when marked on the N. D. will lie to the left and below the points on the sides of the N. P. of $H = 0$ which corresponds in shape to the N. P. of $f = 0$ (for all ρ 's and ξ 's greater than zero).

COROLLARY 2. For every side of the N. P. of $f = 0$ (where all the ρ 's and ξ 's are greater than zero) there will correspond one and only one side of the N. P. of $H = 0$.

Note: These sides will not, of course, be of the same length as the corresponding sides of the N. P. of $f = 0$. This is due to the fact that while a side of the N. P. of $f = 0$ is determined by the points, say, $(\rho_i, k + \alpha_i - \rho_i)$ and $(\rho_j, k + \alpha_j - \rho_j)$, and accordingly is of length

$$\sqrt{(\rho_i - \rho_j)^2 + [(k + \alpha_i - \rho_i) - (k + \alpha_j - \rho_j)]^2}$$

the side of the N. P. of $H = 0$ corresponding to it is determined by the ρ -end points of [111], [11j], [1jj], and [jjj], and therefore will be of length:

$$3\sqrt{(\rho_i - \rho_j)^2 + [(k + \alpha_i - \rho_i) - (k + \alpha_j - \rho_j)]^2}$$

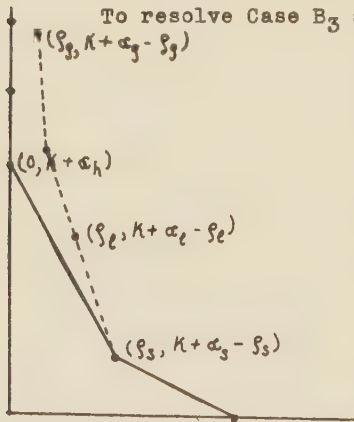
Up to this point we have been considering Case A, i.e., only those ρ 's and ξ 's which are not zero. The results of the consideration may be stated as follows: For every side of the N. P. of $f = 0$, where neither ρ nor ξ is zero, there corresponds one and only one side of the N. P. of $H = 0$, a side which will have the same slope as the corresponding side of the N. P. of $f = 0$, but whose length is three times the length of that corresponding side.

Case B. We shall be dealing in this section with those two sides of the N. P. of $f = 0$ which have been called its $\wp$ -closing side and its ξ -closing side. Let $(\wp_3, k + \alpha_3 - \wp_3)$ be the point on the N. P. of $f = 0$ which with $(0, k + \alpha_h)$ determines the $\wp$ -closing side, and let $(k + \alpha_t - \xi_t, \xi_t)$ be the point which with $(k + \alpha_p, 0)$ determines the ξ -closing side.

Because of the (notational) symmetry noted before we shall have to consider only the $\wp$ -closing side, and draw parallel conclusions for the ξ 's.

There are four situations to be dealt with:

1. When $0 < \wp_3 \leq 2$,
2. When $\wp_3 > 2$, and there exists a $\wp_m = 1$ or 2 , where $k + \alpha_m < k + \alpha_h$,
3. When $\wp_3 > 2$, and there exists a $\wp_{h_j} = 1$ or 2 , and if $\wp_m$ or $\wp_{m_j} = 1$ or 2 , $k + \alpha_m > k + \alpha_h$.
4. When $\wp_3 > 2$, and no $\wp_{h_j} = 1$ or 2 exists, and if $\wp_m$ or $\wp_{m_j} = 1$ or 2 , $k + \alpha_m > k + \alpha_h$.

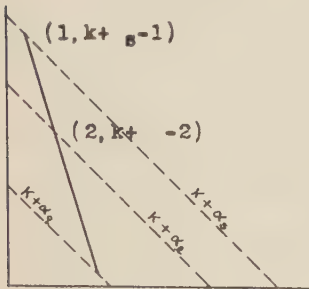


when the first least $\wp$ greater than zero is included.

To resolve Case B₃ and Case B₄ in the simplest fashion, let there be drawn that configuration of lines which shall be termed the Second Newton Polygon of $f = 0$. It shall differ from the ordinary Newton Polygon only in this that all $\wp$'s (or ξ 's as the case may be) equal to zero are to be excluded. This Second Newton Polygon will be considered as closed

Case B₁: When $0 < \rho_s \leq 2$.

a. Let $\rho_s = 1$. The determinants composing $[ssh]$ will give $(0, 3k + 2\alpha_s + \alpha_h - 4)$ as ρ -closing point. A closing point which would lie below this point is possible only for a combination giving an ordinate less than $3k + 2\alpha_s + \alpha_h - 4$. If the ρ of this combination is also 1, say $\rho_m = 1$, it is evident that the degree of the line segment formed will be greater than $3k + 2\alpha_s + \alpha_h - 4$. This is shown from the fact that $(\rho_m, k + \alpha_m - \rho_m)$ is supposed to lie above $(\rho_s, k + \alpha_s - \rho_s)$. Accordingly, $k + \alpha_m > k + \alpha_s$. Therefore $3k + 2\alpha_m + \alpha_h - 4 > 3k + 2\alpha_s + \alpha_h - 4$. Let there exist a $\rho_\ell = 2$, where, as in the figure,



$k + \alpha_q < k + \alpha_\ell < k + \alpha_s$. There can be no question of any combination of $u_{k+\alpha_\ell}$ and $u_{k+\alpha_s}$, i.e., $[X\bar{X}X]$, $[X\bar{X}s]$, $[Xss]$, being concerned since the least possible abscissa that could be produced would be 2--from $[Xss]$ --whereas Theorem (12) holds, and $[sss]$ gives as

ρ -end point: $(1, 3k + 3\alpha_s - 5)$. The determinants $[X\bar{X}h]$ will not produce any point which lies to the left and below the ρ -end points of the lines determined by $[qqq]$, ..., $[sss]$, for if $\rho_\ell = 2$, $k + \alpha_\ell \geq k + \alpha_s - (\alpha_s - \alpha_q)/(\rho_q - 1)$, i.e.,

$$k + \alpha_\ell = k + \alpha_s - (\alpha_s - \alpha_q)/(\rho_q - 1) + \epsilon, \text{ where } \epsilon \geq 0.$$

Accordingly all the points of the line determined by the ρ -end points of $[X\bar{X}h]$ and $[sss]$ will lie to the right and above those of the line determined by the ρ -end points of $[sss]$ and $[qqq]$ if and only if:

$$-\left[\alpha_s - \alpha_h + 2 \frac{\alpha_s - \alpha_q}{\rho_q - 1} - 2\epsilon + 1\right] \leq -\left[\frac{\alpha_s - \alpha_q}{\rho_q - 1} + 1\right].$$

But this inequality resolves into:

$$\rho_E - 1 \geq \frac{\alpha_S - \alpha_E}{\alpha_E - \alpha_S + 2\epsilon}$$

It is evident that the greatest value the right side of this inequality can have is had when $\epsilon = 0$. Then

$$\rho_E - 1 \geq \frac{\alpha_S - \alpha_E}{\alpha_E - \alpha_S}.$$

But this is seen immediately to be the necessary and sufficient condition that $(0, k + \alpha_E)$ should not lie to the left and below that side determined by $(1, k + \alpha_S - 1)$ and $(\rho_E, k + \alpha_E - \rho_E)$. Therefore, $[\chi\chi h]$ will produce no points on the N. D. of $H = 0$ which lie to the left and below the side determined by $[qqq]$, ..., $[sss]$.

Again, no $[mhh]$ when $\rho_m = 2$ will give a ρ -closing point below that of $[ssh]$ when $\rho_S = 1$, for the slope of the side of the N. P. of $f = 0$ determined by $(0, k + \alpha_E)$ and $(1, k + \alpha_S - 1)$ is $-(\alpha_E - \alpha_S + 1)$. Therefore $k + \alpha_m \geq k + 2\alpha_S - \alpha_E$, or

$$k + \alpha_m = k + 2\alpha_S - \alpha_E + \epsilon, \text{ where } \epsilon \geq 0,$$

which means that $[mhh]$ will give as ρ -closing point $(0, 3k + 2\alpha_S + \alpha_E - 4 + \epsilon)$, whereas $[ssh]$ will give as ρ -closing point $(0, 3k + 2\alpha_S + \alpha_E - 4)$.

Accordingly, in no case will $[mhh]$, when $\rho_m = 2$, produce a ρ -closing point below that of $[ssh]$ when $\rho_S = 1$. Therefore, when $\rho_S = 1$, the N. P. of $H = 0$ will be closed with respect to ρ by the ρ -end point of $[ssh]$, viz., at $(0, 3k + 2\alpha_S + \alpha_E - 4)$.

b. Let $\rho_S = 2$. Since every other $\rho_E = 2$ implies that $k + \alpha_E > k + \alpha_S$, it is evident that no $[\chi hh]$ can have a ρ -closing point below that of $[shh]$, viz., $(0, 3k + \alpha_S + 2\alpha_E - 4)$.

No $[mmh]$, when $\rho_m = 1$, can have a ρ -closing point below that of $[shh]$ when $\rho_S = 2$, for the slope of the side of the N. P.

of $f = 0$ determined by $(0, k + \alpha_h)$ and $(2, k + \alpha_s - 2)$ is:

$-(\alpha_h - \alpha_s + 2)/2$. Therefore, $k + \alpha_m \geq k + \alpha_h - (\alpha_h - \alpha_s)/2$ or

$$k + \alpha_m = k + \alpha_h - (\alpha_h - \alpha_s)/2 + \epsilon, \text{ where } \epsilon \geq 0,$$

which means that $[mmh]$ will give as ρ -closing point $(0, 3k + 2\alpha_h + \alpha_s - 4 + 2\epsilon)$, whereas $[shh]$ will give as ρ -closing point $(0, 3k + 2\alpha_h + \alpha_s - 4)$.

Accordingly, in no case will $[mmh]$ when $\rho_m = 1$ produce a ρ -closing point below that of $[shh]$ where $\rho_s = 2$. Therefore, when $\rho_s = 2$, the N. P. of $H = 0$ will be closed with respect to ρ by the ρ -end point of $[shh]$, viz., at $(0, 3k + \alpha_s + 2\alpha_h - 4)$.

Consider the special case when $k + \alpha_h = k + \alpha_n$. It is seen that $[ihh] \equiv [inn]$ develops into

$$\begin{aligned} & x \rho_i + \rho_{np} - 2 \quad y \xi_i + \xi_{nq} - 2 \quad \varphi (3k + \alpha_i + 2\alpha_n - \rho_i - \rho_{np} - \xi_i - \xi_{nq}) \\ & - x \rho_i + 2\rho_{nj} - 2 \quad y \xi_i + 2\xi_{nl} - 2 \quad \varphi (3k + \alpha_i + 2\alpha_n - \rho_i - 2\rho_{nj} - \xi_i - 2\xi_{nl}). \end{aligned}$$

If $\rho_{nj} = 1$, $[inn]$ will have as ρ -end point $(\rho_i, 3k + \alpha_i + 2\alpha_n - \rho_i - 4)$ and if $\rho_{nj} = \rho_{np} \geq 2$, $[inn]$ will have as ρ -end point $(\rho_i + \rho_{np} - 2, 3k + \alpha_i + 2\alpha_n - \rho_i - \rho_{np} - 2)$.

Now if $\rho_s = 2$, (1) if there is a ρ , say ρ_m , whose value is unity, where $k + \alpha_s < k + \alpha_m$, the slope of the line determined by the ρ -end points of $[sss]$ and $[ssn]$ is greater than the slope of the line determined by the ρ -end points of $[sss]$ and $[mmn]$, i.e.,

$$-\left[\frac{\alpha_n - \alpha_s}{2} + 1\right] > -\left[\frac{\alpha_n + 2\alpha_m - 3\alpha_s}{4} + 1\right].$$

Therefore, the line determined by the ρ -end points of $[sss]$ and ssn will be a side of the N. P. of $H = 0$, and the line determined by the ρ -end points of $[ssn]$ and $[mmn]$ will be the ρ -closing side.

(2) If no $\rho_m = 1$ exists, the N. P. of $H = 0$ will be completed with respect to ρ by the side determined by the ρ -end points of $[sss]$ and $[ssn]$ at $(2, 3k + 2\alpha_s + \alpha_n - 6)$, and accordingly $H = 0$ will be composite with respect to x , being composed of $x^2 = 0$ and a curve, non-composite with respect to x of degree $3k + 2\alpha_s + \alpha_n - 6$.

Case B₂. In this case $\rho_s > 2$, and there exists a $\rho_m = 1$ or 2, where $k + \alpha_m < k + \alpha_e$. The conditions impose the following inequality: $k + \alpha_m > k + \alpha_s$. Therefore no point of $[mhh]$, when $\rho_m = 2$, will lie to the left and below the line joining the ρ -end points of $[sss]$ and $[shh]$.

Again, no point of $[mmh]$, when $\rho_m = 1$, will lie to the left and below this same line if the slope of the line joining $[sss]$ and $[shh]$, is greater than or equal to the slope of the line joining $[sss]$ and $[mmh]$. This condition resolves itself into

$$\frac{\alpha_e - \alpha_s}{\alpha_e - \alpha_m} \geq \rho_s$$

But the above inequality is identical with the condition that $(1, k + \alpha_m - 1)$ should not lie to the left and below the line joining $(\rho_s, k + \alpha_s - \rho_s)$ and $(0, k + \alpha_e)$, which of course is assumed.

We have, then, after excluding all other possibilities, reached the conclusion that the line of ρ -end points produced by $[sss]$, $[ssh]$, $[shh]$, will be a side of the N. P. of $H = 0$ when $\rho_s \geq 1$. The upper extreme of this side is $(\rho_s - 2, 3k + \alpha_s + 2\alpha_e - \rho_s - 2)$, and the side has as slope: $-(\alpha_e - \alpha_s + \rho_s)/\rho_s$.

When $\rho_m = 1$, and $k + \alpha_m < k + \alpha_e$, the N. P. of $H = 0$ will be closed by the ρ -end point of $[mmh]$, viz., at $(0, 3k + 2\alpha_m + \alpha_e - 4)$.

When $\rho_m = 2$ and $k + \alpha_m < k + \alpha_e$, the N. P. of $H = 0$ will be closed by the ρ -end point of $[mhh]$, viz., at $(0, 3k + \alpha_m + 2\alpha_e - 4)$.

When $k + \alpha_e = k + \alpha_m$ when $\rho_m = 2$, and (1) if there is a ρ , say ρ_e , whose value is unity, where $k + \alpha_m < k + \alpha_e < k + \alpha_n$, the situation is similar to that under Case B_1 and similar conclusions are drawn, viz., the line determined by the ρ -end points of $[mmm]$ and $[mmn]$ will be a side of the N. P. of $H = 0$, and the line determined by the ρ -end points of $[mmn]$ and $[X\bar{X}n]$ will be the ρ -closing side.

(2) When no $\rho_e = 1$ exists, the N. P. of $H = 0$ will be completed with respect to ρ by the side determined by $[mmm]$ and $[mmn]$ at $(2, 3k + 2\alpha_m + \alpha_n - 6)$, and accordingly $H = 0$ will be composite with respect to x , being composed of $x^2 = 0$ and a curve, proper with respect to x of degree $3k + 2\alpha_m + \alpha_n - 6$.

Case B_3 . In this case we have $\rho_s > 2$, and there exists a $\rho_{e_i} = 1$ or 2, and if ρ_m or $\rho_{m_i} = 1$ or 2, $k + \alpha_m > k + \alpha_e$.

No point produced by $[hhh]$ will lie to the left and below any point of the line joining the ρ -end points of $[sss]$ and $[shh]$, for the slope of the line joining the ρ -end point of $[shh]$ to that of $[sss]$ is greater than the slope of the line joining the ρ -end points of $[sss]$ and $[hhh]$, i.e.,

$$-\left[\frac{3(\alpha_e - \alpha_s)}{3\rho_s - 2} + 1\right] < -\left[\frac{\alpha_e - \alpha_s}{\rho_s} + 1\right].$$

Therefore in this case (B_3) the N. P. of $H = 0$ is closed with respect to ρ by the ρ -end point of $[hhh]$, viz., at $(0, 3k + 3\alpha_e - 4)$.

When $k + \alpha_e = k + \alpha_m$, $[hhh]$ vanishes by definition.

When $k + \alpha_s < k + \alpha_m$, let the Second Newton Polygon of $f = 0$ be drawn. Beginning with $(\rho_s, k + \alpha_s - \rho_s)$, suppose the

first point encountered as we go upwards and to the left is $(\rho_L, k + \alpha_L - \rho_L)$ where $\rho_L > 2$. Then, no determinant, by Theorem (12) will produce any points which lie to the left and below the line joining the ρ -end points of $[sss]$, ..., $[xxx]$.

Further, the slope of the line joining the ρ -end points of $[ssn]$ and $[xxn]$ is exactly that of the line joining $(\rho_s, k + \alpha_s - \rho_s)$ and $(\rho_L, k + \alpha_L - \rho_L)$ of the N. P. of $f = 0$. It is, accordingly, less than the slope of the line joining $[sss]$ and $[ssn]$. But this last slope is identically that of the line joining $(\rho_s, k + \alpha_s - \rho_s)$ and $(0, k + \alpha_n)$. Therefore, the line determined by the ρ -end points of $[ssn]$ and $[xxn]$, where $\rho_L > 2$, will be a side of the N. P. of $H = 0$.

If the next point encountered is ρ_m , where $\rho_m > 2$, the line determined by the ρ -end points of $[x/n]$ and $[mmn]$ will likewise be a side of the N. P. of $H = 0$.

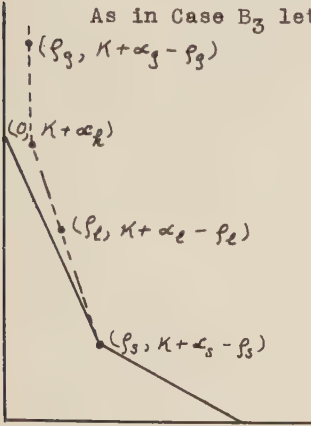
If $(\rho_L, k + \alpha_L - \rho_L)$, $(\rho_m, k + \alpha_m - \rho_m)$, ..., $(\rho_r, k + \alpha_r - \rho_r)$, $(\rho_g, k + \alpha_g - \rho_g)$ are the points encountered in order, the N. P. of $H = 0$ will include the sides determined by the ρ -end points of $[sss]$ and $[ssn]$, $[ssn]$ and $[xxn]$, ..., $[rrn]$ and $[ggn]$, $[ggn]$ and $[gnn]$. Since ρ_g is supposed greater than two, $H = 0$ will be composite with respect to x , being composed of $x^{\rho_g} = 0$ and a curve, proper with respect to x of degree $3k + \alpha_g + 2\alpha_n - \rho_g - 4$.

When $k + \alpha_s = k + \alpha_{n-1}$, not only $[hhh]$ but $[shh]$ vanishes by definition. Accordingly, $[ssh] \equiv [ssh]$, which has as ρ -end point $(2\rho_{n-1} - 2, 3n - 4 - 2\rho_{n-1})$, will be the determinant of highest degree concerned. Therefore, the N. P. of $H = 0$ will be closed with respect to ρ by the line determined by the ρ -end points of $[sss]$ and $[ssn]$ if and only if $\rho_s \equiv \rho_{n-1} = 1$. Otherwise $H = 0$ will be composite, being composed of $x^{2\rho_{n-1}-2} = 0$ and

curve proper with respect to x , of degree $3n - 4 - 2\rho_{n-1}$.

Case B₄. In this case $\rho_s > 2$, and no $\rho_{hj} = 1$ or 2 exists, and if ρ_m or $\rho_{mj} = 1$ or 2, $k + \alpha_k > k + \alpha_k$.

Let the Second Newton Polygon be drawn.



As in Case B₃ let us begin with $(\rho_s, k + \alpha_s - \rho_s)$ and let us suppose that the first point encountered, as we go upwards and to the left, is $(\rho_e, k + \alpha_e - \rho_e)$ where $\rho_e \geq 2$. Then no determinant, by the Theorem, will produce any points which lie to the left and below the line joining the ρ -end points of $[sss]$, ..., $[xxx]$.

Further, the upper extreme of the line joining the ρ -end points of $[xxx]$, $[xhx]$, $[xhh]$ will be $(\rho_e - 2, 3k + \alpha_e + 2\alpha_k - \rho_e - 2)$, i.e., the ρ -end point of $[xhh]$.

Now the slope of the line joining the ρ -end points of $[shh]$ and $[xhh]$, i.e., the slope of the line joining $(\rho_e - 2, 3k + \alpha_e + 2\alpha_k - \rho_e - 2)$ and $(\rho_s - 2, 3k + \alpha_s + 2\alpha_k - \rho_s - 2)$ is exactly that of the line joining $(\rho_s, k + \alpha_s - \rho_s)$ and $(\rho_e, k + \alpha_e - \rho_e)$ of the N. P. of $f = 0$. It is, accordingly, less than the slope of the line joining $[sss]$ and $[shh]$. But this last slope is identically that of the line joining $(\rho_s, k + \alpha_s - \rho_s)$ and $(0, k + \alpha_k)$.

Therefore the line determined by the ρ -end points of $[shh]$ and $[xhh]$ where $\rho_e \geq 2$ will be a side of the N. P. of $H = 0$.

Note: If $\rho_e = 2$, the N. P. of $H = 0$ is closed at $(0, 3k + \alpha_e + 2\alpha_k - 4)$. If $\rho_e = 1$, the N. P. of $H = 0$ is closed at $(0, 3k + 2\alpha_e + \alpha_k - 4)$.

However, suppose that $\rho_e > 2$, and let there be a point of the Second Newton Polygon of $f = 0$, say $(\rho_e, k + \alpha_e - \rho_e)$, where $\rho_e \geq 2$, which is next encountered as we proceed upwards and to the left. Then similar arguments may be used to show that the line determined by the ρ -end points of $[\lambda hh]$ and $[qhh]$ where $\rho_e \geq 2$ will be a side of the N. P. of $H = 0$. Continue this process until the smallest ρ greater than zero is reached. Suppose this is ρ_f . Then if

1. $\rho_f = 1$, the N. P. of $H = 0$ will be closed at $(0, 3k + 2\alpha_g + \alpha_e - 4)$.

2. $\rho_f = 2$, the N. P. of $H = 0$ will be closed at $(0, 3k + \alpha_g + 2\alpha_e - 4)$.

3. $\rho_f > 2$, the N. P. of $H = 0$ will not be closed with respect to ρ , but its extreme upper point will be $(\rho_f - 2, 3k + \alpha_g + 2\alpha_e - \rho_f - 2)$.

Accordingly, if (1) or (2) holds $H = 0$ will be a proper curve with respect to λ , but if (3) holds $H = 0$ will be a composite curve with respect to λ , being composed of $x^{\rho_f-2} = 0$, and $H_1 = 0$ a proper curve.

The only difference that $k + \alpha_e = k + \alpha_n$ makes is that there can exist no $\rho_f \leq 2$ under the conditions of this case. Accordingly, $H = 0$ will always be composite with respect to x , being composed of $x^{\rho_f + \rho_n p - 2} = 0$ and a curve proper with respect to x of degree $3k + \alpha_g + 2\alpha_n - \rho_f - \rho_n p - 2$.

IX. THE RELATIONSHIP OF THE LINE AT INFINITY TO THE HESSIAN

Let $3n - 6 - \lambda$ be the degree of the terms of highest degree produced by the determinants of (6) as s takes all its admissible values. Then $H(xyz) = 0$ will be factorable into $z^\lambda = 0$ and a polynomial in x, y, z which does not vanish for $z = 0$.

Accordingly, the line at infinity, $z = 0$, is repeated λ times in the Hessian Curve.

X. COMPOSITION OF THE N. P. OF $H = 0$

The Newton Polygon of $H = 0$ at a k -fold point ($1 < k < n$) of $f = 0$ will be composed of the following sides:

I. When $k + \alpha_L < k + \alpha_k$.

1. All those sides which correspond in number and slope to the sides of the N. P. of $f = 0$, for which both ρ 's are greater than zero.

2. In Case B_1 , if $\rho_s = 1$ the side determined by the ρ -end points of $[sss]$ and $[ssh]$, or if $\rho_s = 2$ the side determined by the ρ -end points of $[sss]$ and $[shh]$.

In Case B_2 , (a) the side determined by the ρ -end points of $[sss]$ and $[shh]$, (b) together with the side determined by $[shh]$ and $[mmh]$ if $\rho_m = 1$, or the side determined by $[shh]$ and $[mhh]$ if $\rho_m = 2$.

In Case B_3 , (a) the side determined by the ρ -end points of $[sss]$ and $[shh]$, (b) together with the side determined by the ρ -end points of $[shh]$ and $[hhh]$.

In Case B_4 , (a) the side determined by the ρ -end points of $[sss]$ and $[shh]$, (b) together with all those sides which correspond in number and slope to the sides of the Second Newton Polygon of $f = 0$. These sides are determined by the ρ -end points of $[shh]$ and $[\chi hh]$, $[\chi hh]$ and $[qhh]$, ..., $[fnh]$ and $[ghh]$ (or $[ggh]$ according as $\rho_f = 2$ or 1), where $\rho_e, \rho_f, \dots, \rho_f$ are taken as they are encountered, and where ρ_f is the first least ρ greater than zero which is encountered.

3. That group of sides which bears the same relationship to the ξ 's which Case B bears to the ρ 's.

II. When $k + \alpha_h = k + \alpha_n$.

1. All those sides which correspond in number and slope to the sides of the N. P. of $f = 0$, for which both ρ 's are greater than zero.

2. In Case B_1 , if $\rho_s = 1$, the side determined by the ρ -end points of $[sss]$ and $[ssn]$, or if $\rho_s = 2$ and (1) if there is a $\rho_m = 1$, where $k + \alpha_s < k + \alpha_m$, the side determined by the ρ -end points of $[sss]$ and $[ssn]$ together with that determined by the ρ -end points of $[ssn]$ and $[mnn]$. (2) If no $\rho_m = 1$ exists the side determined by the ρ -end points of $[sss]$ and $[ssn]$.

In Case B_2 , (a) the side determined by the ρ -end points of $[sss]$ and $[ssn]$, (b) together with the side determined by $[ssn]$ and $[mnn]$, and (1) when also a $\rho_l = 1$ exists where $k + \alpha_m < k + \alpha_l$, the side determined by $[mnn]$ and $[l\lambda n]$.

In Case B_3 , those sides corresponding to the Second Newton Polygon of $f = 0$, determined by the ρ -end points of $[sss]$ and $[ssn]$, $[ssn]$ and $[l\lambda n]$, ..., $[rrn]$ and $[ggn]$; together with $[ggn]$ and $[gnn]$ except when $k + \alpha_g = k + \alpha_{n-1}$.

Case B_4 , same as Case B_3 .

XI. SUBSTITUTIONS WHICH LEAD TO THE PUISEUX EXPANSIONS BY WHICH THE BRANCHES OF $H = 0$ ARE SEPARATED

In the following substitutions the numbering and lettering will be that of Section X.

I. When $k + \alpha_h < k + \alpha_n$.

1. Of the type

$$x = t^3(\alpha_j - \alpha_i + \rho_i - \rho_j)$$

$$y = ut^3(\rho_i - \rho_j)$$

2. Case B₁,

$$\begin{array}{ll} x = t^{\alpha_h - \alpha_s + \rho_s} & \text{or} \\ y = ut^{\rho_s} & x = t^{2(\alpha_h - \alpha_s + \rho_s)} \\ & y = ut^{2\rho_s} \end{array}$$

Case B₂,

$$\begin{array}{ll} (a) & x = t^{2(\alpha_h - \alpha_s + \rho_s)} \\ & y = ut^{2\rho_s} \\ (b) & x = t^{2\alpha_m - \alpha_h - \alpha_s + \rho_s - 2\rho_m} \quad \text{or} \quad x = t^{\alpha_m - \alpha_s + \rho_s - \rho_m} \\ & y = ut^{\rho_s - 2\rho_m} \quad y = ut^{\rho_s - \rho_m} \end{array}$$

Case B₃,

$$\begin{array}{ll} (a) & x = t^{2(\alpha_h - \alpha_s + \rho_s)} \\ & y = ut^{2\rho_s} \\ (b) & x = t^{\alpha_h - \alpha_s + \rho_s - 2} \\ & y = ut^{\rho_s - 2} \end{array}$$

Case B₄,

$$\begin{array}{ll} (a) & x = t^{2(\alpha_h - \alpha_s + \rho_s)} \\ & y = ut^{2\rho_s} \\ (b) & \text{of the type } x = t^{\alpha_m - \alpha_\ell + \rho_\ell - \rho_m} \\ & y = ut^{\rho_\ell - \rho_m} \end{array}$$

3. Of the type

$$\begin{array}{l} x = t^{\xi_\tau} \\ y = ut^{\alpha_p - \alpha_\tau + \xi_\tau}, \text{ etc.} \end{array}$$

II. When $k + \alpha_h = k + \alpha_n$.

1. Of the type

$$\begin{array}{l} x = t^{3(\alpha_j - \alpha_i + \rho_i - \rho_j)} \\ y = ut^{3(\rho_i - \rho_j)} \end{array}$$

2. Case B₁

$$x = t^{\alpha_n - \alpha_s + \rho_s}$$

$$y = ut^{\rho_s}$$

$$\text{or (1) } x = t^{\alpha_n - \alpha_s + \rho_s}$$

$$y = ut^{\rho_s}$$

and

$$x = t^{2(\alpha_m - \alpha_s + \rho_s - \rho_m)}$$

$$y = ut^{2(\rho_s - \rho_m)}$$

$$(2) \quad x = t^{\alpha_n - \alpha_s + \rho_s}$$

$$y = ut^{\rho_s}$$

Case B₂

$$(a) \quad x = t^{\alpha_n - \alpha_s + \rho_s}$$

$$y = ut^{\rho_s}$$

and

$$(b) \quad x = t^{2(\alpha_m - \alpha_s + \rho_s - \rho_m)}$$

$$y = ut^{2(\rho_s - \rho_m)}$$

and if also a $\rho_\ell = 1$ exists where $k + \alpha_m < k + \alpha_\ell$,

$$x = t^{2(\alpha_\ell - \alpha_m + \rho_m - \rho_\ell)}$$

$$y = ut^{2(\rho_m - \rho_\ell)}$$

Case B₃

$$x = t^{\alpha_n - \alpha_s + \rho_s}$$

$$y = ut^{\rho_s}$$

$$x = t^{2(\alpha_\ell - \alpha_s + \rho_s - \rho_\ell)}$$

$$y = ut^{2(\rho_s - \rho_\ell)}$$

and those of the type

$$x = t^{2(\alpha_m - \alpha_\ell + \rho_\ell - \rho_m)}$$

$$y = ut^{2(\rho_\ell - \rho_m)}$$

and finally

$$x = t^{\alpha_n - \alpha_j + \rho_j - 2}$$

$$y = ut^{\rho_j - 2}$$

except when $k + \alpha_g = k + \alpha_{n-1}$.

Case B₄

$$x = t^{\alpha_n - \alpha_s + \rho_s}$$

$$y = ut^{\rho_s}$$

$$x = t^{2(\alpha_\ell - \alpha_s + \rho_s - \rho_\ell)}$$

$$y = ut^{2(\rho_s - \rho_\ell)}$$

and those of the type

$$x = t^{2(\alpha_m - \alpha_l + \rho_l - \rho_m)}$$

$$y = ut^{2(\rho_l - \rho_m)}$$

and finally

$$x = t^{\alpha_n - \alpha_g + \rho_g - \rho_{np}}$$

$$y = ut^{\rho_g - \rho_{np}}$$

3. Of the type

$$x = t^{\frac{1}{2}\tau}$$

$$y = ut^{\alpha_p - \alpha_r + \frac{1}{2}\tau}, \text{ etc.}$$

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VITA

The Reverend James Edward Case S. J. was born at Odin, Illinois, July 31, 1896. He attended the Parochial Schools at Springfield, Illinois. He made his High School studies at Campion, Prairie du Chien, Wisconsin from 1908-1912, and one year of college at Campion from 1913-1914.

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